Chapter 2

TOP OR BOTTOM: WHICH ONE MATTERS?

Helping Students Reason About Generalizations Regarding Numerators and Denominators

Strategy #2
Provide opportunities for students to investigate, assess, and refine mathematical "rules" and generalizations.

From Principles and Standards for School Mathematics

Number and Operations Standard: Grades 3–5:

Through the study of various meanings and models of fractions—how fractions are related to each other and to the unit whole and how they are represented—students can gain facility in comparing fractions, often by using benchmarks such as $\frac{1}{2}$ or 1.

Classroom Scenario

In order to help her students understand the relative size of different fractions, Ms. Alvarez posed the following question, "Would you rather share your favorite pizza with three other people or seven other people? If the group ate the whole pizza, and everyone ate the same amount so they each got a fair share, which way would you get more: with three other people or seven other people? Work together in your group and make a poster showing your answer. Provide a clear explanation using pictures, numbers, and words."

Ms. Alvarez was very pleased by her students' posters and presentations. For the most part, the students' posters indicated that they would much rather share the pizza with three other people because that way they would get more pizza. (See Figure 2–1.)

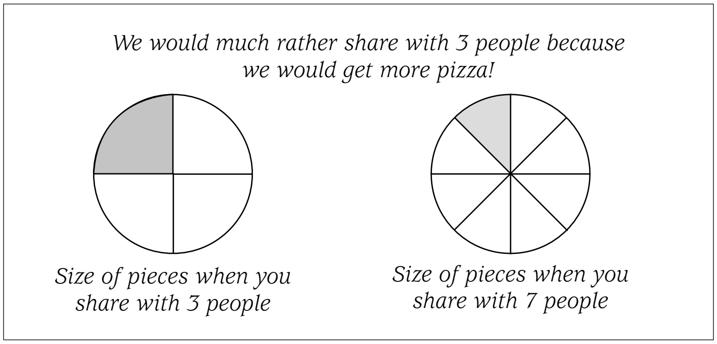


Figure 2–1 Poster Example

After reaching consensus that everyone would rather be in the group with four people than with eight, because having fewer people to share with would mean each person would get more pizza, Ms. Alvarez posed the following question: "So now I have a question for all of you. Which is bigger, one-fourth or one-eighth? Give me a thumbs-up if you think one-fourth is bigger, give me a thumbs sideways if you think one-eighth is bigger, and put your thumb on your nose if you're not sure."

Ms. Alvarez knew that this question would present a challenge for many students even though they all understood the sharing context. As she looked around her classroom she saw that while several students had their thumbs up, a fair number had them sideways or on their nose. Ms. Alvarez asked her students to discuss the question with their table groups and to try to come up with a generalization about fraction sizes in general.

"Well," Dominique began, "I think you could say that if the number of pieces the thing is cut into is smaller, the fraction is larger. And if the number of pieces the thing is cut into is larger, the fraction is smaller."

"Thank you, Dominique. Can someone restate Dominique's idea using the math word that means 'the number of pieces the thing is cut into'?" asked Ms. Alvarez.

Noticing Noah's hand, Ms. Alvarez called on him. "Here goes. The smaller the denominator, the larger the fraction. The larger the denominator, the smaller the fraction."

"What do you think about that?" Ms. Alvarez asked the class. Several students nodded enthusiastically and many gave Noah a thumbs-up. "Let's all say Noah's idea together: *The smaller the denominator, the larger the fraction. The larger the denominator, the smaller the fraction*." Ms. Alvarez recorded Noah's conjecture while chanting along with the class, happy that they seemed to understand this important idea.

A few days later Ms. Alvarez was looking over the students' homework and was dismayed to see that in response to the prompt

Circle the larger fraction: $\frac{1}{2}$ $\qquad$ $\frac{3}{4}$

many of them had indicated that $\frac{1}{2}$ was greater than $\frac{3}{4}$. When Ms. Alvarez asked Dominique to explain her answer during homework check, Dominique replied, "Remember you taught us last week—the smaller the denominator, the larger the fraction. The larger the denominator, the smaller the fraction. Two is smaller than four, so one-half has got to be bigger than three-fourths."

As several students nodded their agreement with Dominique, Ms. Alvarez realized they had overgeneralized this important idea. She also realized her students were thinking only of the denominators of the fractions they were comparing, not realizing that in order to understand the value of a fraction they needed to consider both the numerator and denominator as well as their relationship to each other. Ms. Alvarez decided she would need to review this with her students and help them understand that both parts of a fraction are equally important.

What's the Math?

When comparing fractions, students need to consider the size of the wholes and interpret each fraction as a single number defined by the relationship between the numerator and the denominator.

Fractions can be compared in several different ways. Finding a common denominator or cross-multiplying to compare are two common approaches that students learn in school, but these approaches do not explicitly require a consideration of the size of the fractions. As Van de Walle, Karp, and Bay-Williams (2009) note, other strategies include:

- More of the same-sized parts (same denominators);
- Same number of parts but parts of different sizes (same numerators);
- More and less than one-half or one whole; and
- Closeness to one-half or one whole.

These strategies may support reasoning about the size of the fractions; however, they may not work with all fractions. Here are examples of these and other comparison strategies:

Finding a Common Denominator

$$\frac{5}{6} \times \frac{4}{4} = \frac{20}{24}$$

$$\frac{7}{8} \times \frac{3}{3} = \frac{21}{24}$$

$$\frac{20}{24} < \frac{21}{24}, \text{ so } \frac{5}{6} < \frac{7}{8}$$

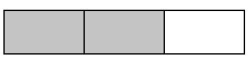
Cross-Multiplication

$$\frac{5}{6} \quad \frac{7}{8}$$

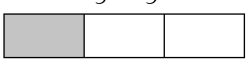
$$5 \times 8 = 40 \qquad 6 \times 7 = 42$$

$$40 < 42, \text{ so } \frac{5}{6} < \frac{7}{8}$$

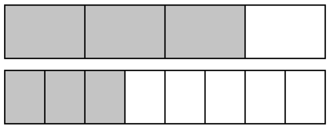
More of the Same-Size Parts



$$\frac{2}{3} > \frac{1}{3}$$



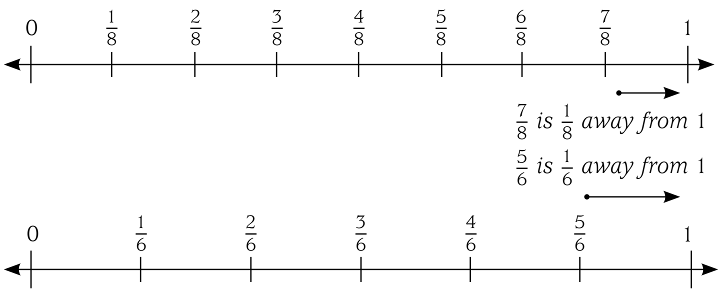
Same Number of Parts But Parts of Different Sizes



Fourths are larger then eighths, so $\frac{3}{4} > \frac{3}{8}$

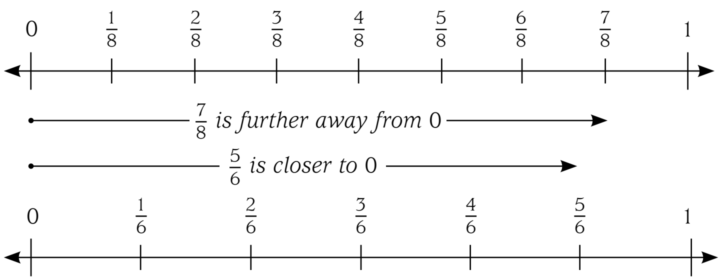
Closeness to One Whole

$\frac{7}{8}$ *is* $\frac{1}{8}$ *less than* 1. $\frac{5}{6}$ *is* $\frac{1}{6}$ *less than* 1. $\frac{1}{6}$ *is greater than* $\frac{1}{8}$, *so* $\frac{7}{8}$ *is closer to* 1.



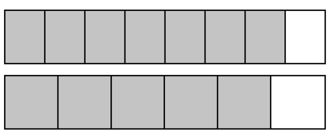
Distance from Zero

One could locate both numbers on a number line to determine distance from 0 *(alternatively, a single number line could be used):*



Size of the Shaded Area

$\frac{7}{8} > \frac{5}{6}$



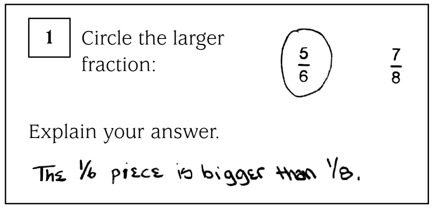
What's the Research?

During the spring of 2009, we asked 267 fourth and sixth graders to respond to the following prompt:

> *Circle the larger fraction.* $\frac{5}{6}$ $\frac{7}{8}$
> *Explain your answer.*

We had hoped that students would select $\frac{7}{8}$ and explain their answer using a variety of strategies such as the ones described in the "What's the Math?" section.

We were very surprised by the number of students who chose $\frac{5}{6}$ as the larger of the two fractions (40 percent of fourth graders and 34 percent of sixth graders). Even more surprising were the reasons students gave for *why* they circled $\frac{5}{6}$. A large percentage of the students who indicated that $\frac{5}{6}$ is larger than $\frac{7}{8}$ explained their answer by stating that "sixths are bigger than eighths," and many


1 Circle the larger fraction: $\frac{5}{6}$ $\frac{7}{8}$

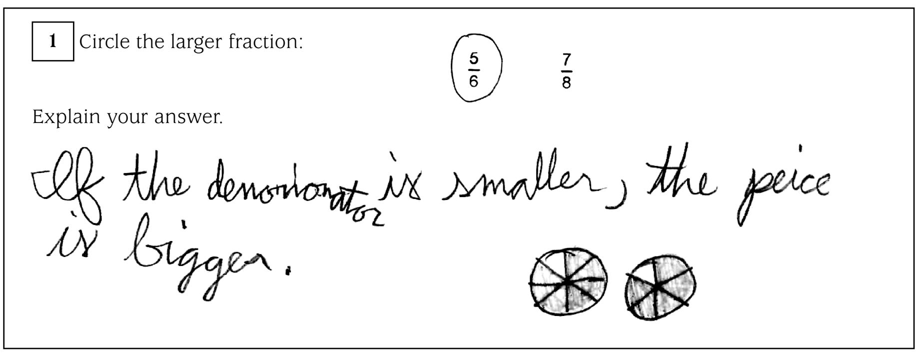
Explain your answer.

The 1/6 piece is bigger than 1/8.

Figure 2–3 The $\frac{1}{6}$ piece is bigger than $\frac{1}{8}$.

wrote a variation on "the smaller the denominator, the larger the fraction." For example, one fourth grader represented $\frac{5}{6}$ and $\frac{7}{8}$ using an area model (a circle) and explained that "if the denominator is smaller the peice [*sic*] is bigger." (See Figure 2–2.)

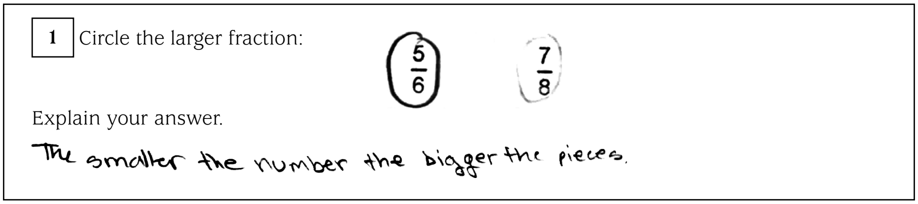
We saw very similar responses among the sixth graders who chose $\frac{5}{6}$ as the larger fraction. For example, one student made a comparison in terms of the specific fractions included in the problem, sixths and eighths. (See Figure 2–3.) Another sixth grader


1 Circle the larger fraction: $\frac{5}{6}$ $\frac{7}{8}$

Explain your answer.

If the denominator is smaller, the peice is bigger.

Figure 2–2 If the denominator is smaller, the piece is bigger.


1 Circle the larger fraction: $\frac{5}{6}$ $\frac{7}{8}$

Explain your answer.

The smaller the number the bigger the pieces.

Figure 2–4 The smaller the number the bigger the pieces.

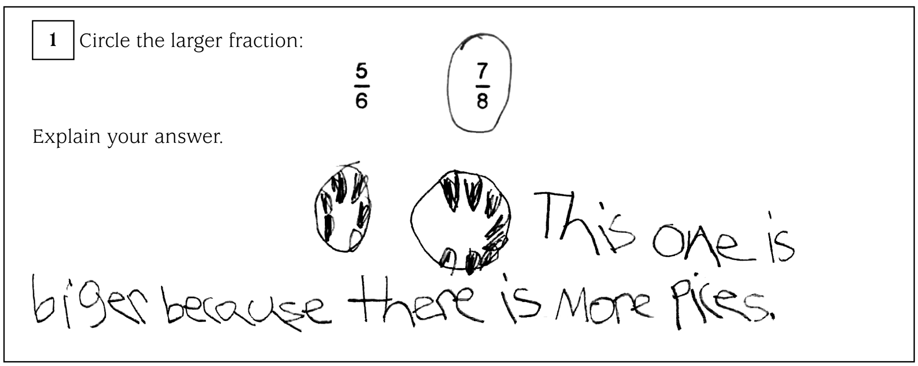
provided a more generalized rationale, that "the smaller the number the bigger the pieces." (See Figure 2–4.)

This finding indicates that students may be responding to only one aspect of the written fraction, in this case the denominator, instead of attending to both the numerator and denominator as necessary components of the number. This is like looking at a number such as 432, where each of the digits provides crucial information about the value of the number, but the digits have to be considered in relation to each other to provide the whole story. If students focus only on the numerator or denominator, they are not considering the value of the fraction.

Even among students who circled the correct answer ($\frac{7}{8}$) some indicated that their reason for choosing $\frac{7}{8}$ was based on the value of the numerator only. (See Figure 2–5.)

Still other students paid attention to both the numerators and denominators when comparing the fractions, but did not consider the relationship between them. For instance, a subset of students indicated that $\frac{7}{8}$ is the larger fraction because 7 is greater than 5 and 8 is greater than 6. (See Figure 2–6.)

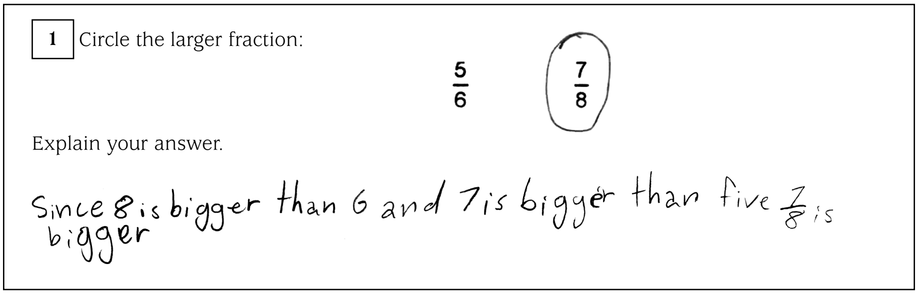
These types of responses present compelling evidence that many


1 Circle the larger fraction: $\frac{5}{6}$ $\frac{7}{8}$

Explain your answer.

This one is biger because there is More Pices.

Figure 2–5 This one is bigger because there are more pieces.


1 Circle the larger fraction:

$\frac{5}{6}$ $\frac{7}{8}$

Explain your answer.

Since 8 is bigger than 6 and 7 is bigger than five $\frac{7}{8}$ is bigger

Figure 2–6 Since 8 is bigger than 6 and 7 is bigger than five, $\frac{7}{8}$ is bigger.

students are not attending to the relationship between numerators and denominators when asked to reason about fractional values and make comparisons between two or more fractions. This provides further confirmation of the claims made by Nancy Mack, a professor at Grand Valley State University, that students often rely on whole number strategies when solving fraction problems (2001).

When asked to compare two fractions such as $\frac{1}{2}$ and $\frac{3}{4}$, many of Ms. Alvarez's students had overgeneralized the important mathematical idea that fractions with smaller denominators are larger than those with larger denominators. What her students failed to realize was that this generalization or "rule" had developed from a situation where the numerators of both fractions were the same ($\frac{1}{4}$ versus $\frac{1}{6}$). When comparing $\frac{1}{2}$ and $\frac{3}{4}$, many students did not realize or remember that both the numerators and denominators of the fractions needed to be considered when comparing their values. When students overgeneralize in this manner they may not be thinking of fractions as numbers, but only as groups of items or parts of shaded areas. The following classroom activities will help students increase their understanding of fractions as numbers and increase their ability to develop and apply appropriate generalizations.

Classroom Activities

Activity 2.1 *Number Line Activities with Cuisenaire Rods*

Overview

In these activities students use the number line to discover that the rule "the smaller the denominator, the larger the fraction" holds true only when the numerators are the same because $\frac{1}{2}$ is indeed larger than $\frac{1}{4}$, and $\frac{1}{4}$ is larger than $\frac{1}{8}$. In addition, $\frac{2}{2}$ is larger than $\frac{2}{4}$, and $\frac{2}{4}$ is larger than $\frac{2}{8}$. Students can also see, however, that when the numerators are different, the comparison of any two fractions is more complex. Knowing the denominator of a fraction is a very important aspect of understanding its value, but it is only part of the information that is necessary. In much the same way, knowing only the numerator of a fraction is not sufficient when it comes to understanding the value.

1. Pass out the *12-cm Number Lines* recording sheets (Reproducible 2a). Using 12-cm as the unit interval allows students to use the rods to partition the unit into twelfths, sixths, fourths, thirds, and halves.

Materials

12-cm Number Lines *recording sheet, 1 copy per student (Reproducible 2a)*

Cuisenaire rods, 1 set per pair of students

Manipulative Note

Cuisenaire rods are wooden or plastic blocks that range in length from 1 to 10 centimeters. Each rod of a given length is the same color. That is, all of the 1-cm rods are white, all of the 2-cm rods are red, all of the 3-cm rods are light green, and so on.

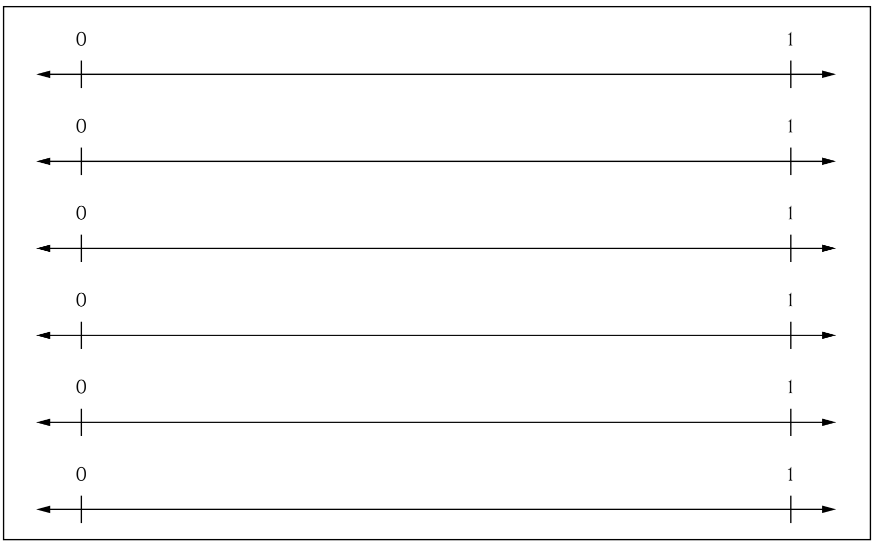
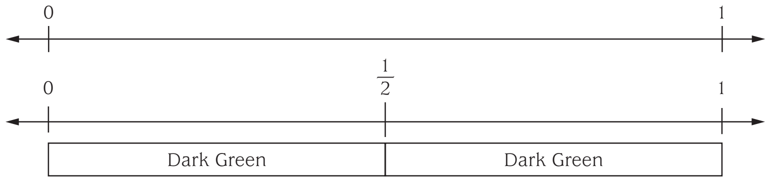
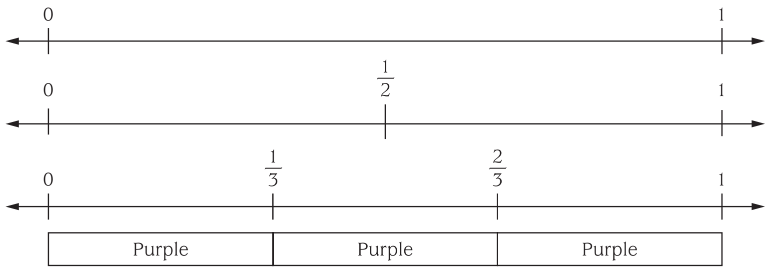


Figure 2–7 12-cm Number Lines (Reproducible 2a)

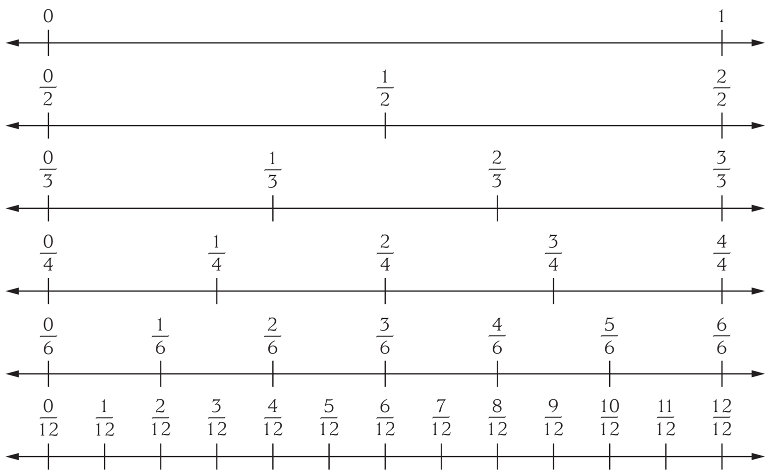
2. Instruct students to find the rod that allows them to partition the units on the first number line into halves. Have students use the 6-cm dark green rod to mark $\frac{1}{2}$ on the number line:



3. Next, have students find the rod that will enable them to partition the next number line in thirds:



4. Continue in this manner until students have partitioned and labeled the final three number lines into fourths, sixths, and twelfths:



Teaching Note

See Chapter 3 for further discussion of helping students understand fractions as numbers.

5. After students have partitioned and labeled their number lines, ask for observations about the numbers they have written. It is important that you begin to refer to fractions as numbers, so that students become comfortable with this language as well.

6. Pose one or more of the following questions to help students reason about fractions as numbers.

Questions to Help Students Reason About Fractions as Numbers

What number is halfway between zero and one?	Some students may initially be surprised that there are numbers between 0 and 1.
What number is halfway between zero and one-half?	Realizing that $\frac{1}{4}$ lies between 0 and $\frac{1}{2}$ on the number line reinforces the relationship between halves and fourths.
What other numbers are the same as one-half?	Students may initially say there are several numbers here: $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{6}{12}$. This is an excellent opportunity to introduce the idea that although these look like different numbers, they are actually different ways to name the same number, much like "one hundred" can also be called "ten tens." This is also an opportunity to discuss what names for the same number have in common.
What number is one-fourth more than one-half? One-sixth more than one-half?	This question can help students begin to reason about relative value of different fractions and compute without the need for converting to numbers with common denominators.
What number is one-sixth less than one?	This question encourages students to compare fractions to the unit.
What number is one-third more than one?	This question exposes students to fractions greater than one and can support their understanding that $\frac{4}{3}$ is the same as $1\frac{1}{3}$.
What number is halfway between one-twelfth and three-twelfths?	This question provides another chance for students to encounter equivalents. They can also begin to reason why there is no sixth equivalent to $\frac{1}{12}$ or $\frac{3}{12}$ (or $\frac{5}{12}$, $\frac{7}{12}$, $\frac{9}{12}$, or $\frac{11}{12}$).

(*continued*)

Which number is closest to zero?	This provides another example of when "the larger the denominator, the smaller the fraction" is true.
Which number is closest to one?	This can help students see that knowing both the numerator and the denominator is necessary to understanding a fraction's value. It can also provide a very reliable and frequently sufficient way to compare fractions, without needing to find common denominators and create equivalent fractions.
What would you call a number halfway between zero and one-twelfth?	This question asks students to extend their understanding and provides a foundation for helping them reason about fraction multiplication, that is, Why does $\frac{1}{2} \times \frac{1}{12} = \frac{1}{24}$?

Since the materials students used for this activity do not include rods that enable partitioning the number line into twenty-fourths, this last question may initially seem like a trick question to some students. By posing this question you can encourage students to look deeply at the relationships between the other fractions they have been working with. The ensuing discussion can provide many opportunities for students' limited understanding to surface and for students to refine their thinking about fraction relationships.

Any of these questions can be posed during whole-class discussions, given to groups to investigate and report back, or used for assessment purposes. Allow students to discuss their thinking and provide a rationale for their responses. These questions are not designed to be answered with one-word responses and no discussion. Any one of the questions can serve as a springboard for an entire lesson.

Assessment Opportunity

As your students are working with the two-unit number line, don't be surprised if some of them initially identify the 6-cm dark green rods as fourths. This is very common! In fact, these students are correct, in that one 6-cm rod is $\frac{1}{4}$ of the distance from 0 to 2 (24 cm). What they are not considering, however, is that the unit they are partitioning is the distance between 0 and 1, and in that case each 6-cm rod is equal to $\frac{1}{2}$.

Activity 2.1 Extension — *Working with the Two-Unit Number Line*

Overview

To provide opportunities for your students to work with fractions between 1 and 2, use the *Two-Unit Number Lines* recording sheets that are numbered from 0 to 2. (See Figure 2–8 [Reproducibles 2b–2c].)

1. Pass out the *Two-Unit Number Lines* (Reproducibles 2b–2c). Have your students use the Cuisenaire rods as in "Activity 2.1: Number Line Activities with Cuisenaire Rods" to partition the number lines into halves, thirds, fourths, sixths, and twelfths.

2. Ask students questions as before, this time providing opportunities for them to reason about fractions greater than one as well as those less than one.

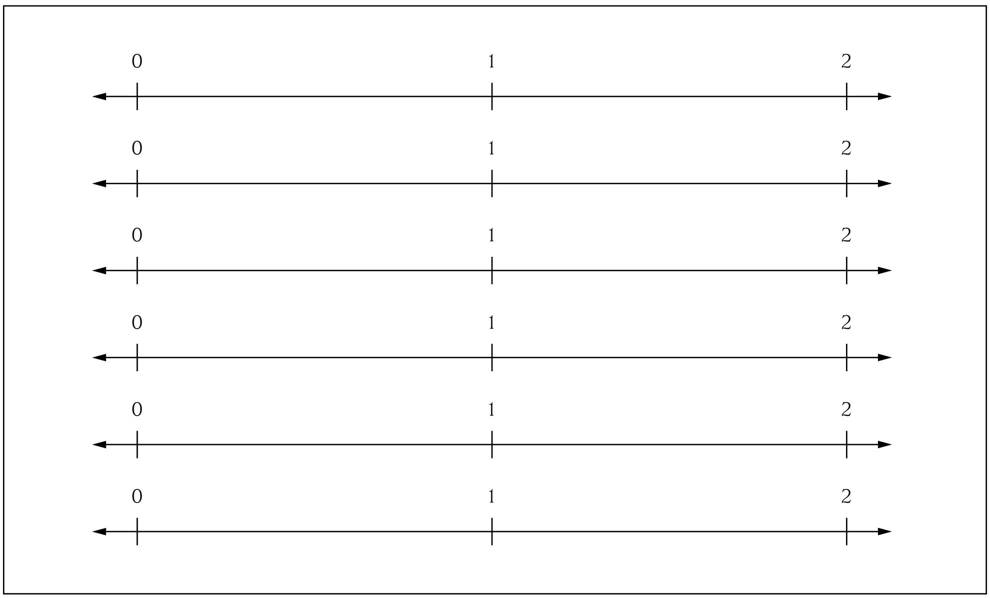


Figure 2–8 Two-Unit Number Lines (Reproducibles 2b–2c)

Activity 2.2 *Is It Always True?*

Overview

This activity will help students to examine and refine their ideas about mathematical "rules." It can also help students avoid making mistakes as they become more aware of common errors.

Materials

student-generated list of mathematical "rules" or scenarios that require students to decide on a solution

Part 1

1. Work with students to name some common rules or generalizations they may have heard about fractions. The list may include some of these ideas:

   - The smaller the denominator, the larger the fraction.
   - The larger the denominator, the smaller the fraction.
   - You can't compare fractions with different denominators.
   - Fractions are always less than 1.
   - To compare two fractions, you only need to look at the numerators (or denominators).
   - Finding a common denominator is the only way to compare fractions with different denominators.

2. Ask students to decide if they think the rules are always true, sometimes true, or never true. Some of the statements that students identify as sometimes true, such as "To compare two fractions, you only need to look at the numerators," can be modified slightly by changing the wording. If the statement is changed to read, "To compare two fractions *with the same denominators*, you only need to look at the numerators," it then becomes a rule that is always true.

Scenario A: J. J.'s Solution

J. J. was solving the following problem:

Circle the larger fraction and explain your answer. $\frac{5}{6}$ $\frac{7}{8}$

J. J. circled $\frac{5}{6}$ and wrote the following explanation: *I know that $\frac{5}{6}$ is larger than $\frac{7}{8}$ because sixths are bigger than eighths. The smaller denominator means the fraction is larger.*

- What do you think of J. J.'s explanation?
- What important idea about fractions did J. J. use to solve the problem?
- Does J. J.'s reasoning make sense? Why or why not?

Part 2

3. Present Scenarios A and B to your class. Ask your class to figure out what the students might have been thinking in each scenario.

   By asking the class to evaluate another student's faulty reasoning, we can help them to become more aware of common pitfalls and thus to avoid making those same kinds of mistakes in the future. The Math Pathways and Pitfalls program developed at WestEd by Carne Barnett-Clarke and Alma Ramirez has had a lot of success with this approach by explicitly focusing students' attention on common misconceptions. Since many of these misconceptions are based on sound mathematical ideas, the more we can help students understand the origins of common misconceptions, the more we can facilitate their development of deep and solid understandings. We can help our students become more reflective and independent learners by encouraging them to frequently ask themselves questions such as, "Does this make sense?" "Is it always true?" "What might a common mistake be?"

Scenario B: Sarah's Solution

On a different comparison task another student, Sarah, solved the problem in this way:

Circle the larger fraction and explain your answer.
$\frac{3}{4}$ $\frac{5}{12}$

Sarah circled $\frac{5}{12}$ and wrote the following explanation: *Five is more pieces than 3 pieces so $\frac{5}{12}$ is more than $\frac{3}{4}$.*

- What do you think of Sarah's explanation?
- What important idea about fractions did Sarah use to solve the problem?
- Does Sarah's reasoning make sense? Why or why not?

Wrapping It Up

As we discussed at the beginning of this chapter, it is not uncommon for children to misapply generalizations as they attempt to make sense of new and complex material. As teachers, we may inadvertently contribute to this natural inclination by teaching our students to memorize so-called "rules," such as "You can't subtract a bigger number from a smaller one," "Rectangles always have two long sides and two short sides," and "The smaller the denominator, the bigger the fraction." Helping students question and refine generalizations and strategies is an extremely valuable exercise that supports their development as mathematical sense-makers. When students overgeneralize or misapply a mathematical rule we can help them refine their thinking by asking questions like these: "Is it always true?" "Under what conditions is it not true?" and "What would you need to change to make it true?" For instance, a generalization such as "The smaller the denominator, the bigger the fraction" could be modified to read, "The smaller the denominator, the bigger the fraction *as long as the numerators are the same*."

Making and refining mathematical generalizations is an important aspect of understanding the connections between important mathematical ideas. As teachers, we strive to help students make sense of the content we are teaching by guiding them to make those generalizations. It is often tempting to simplify mathematical principles and procedures in order to help students be successful. However, to help students build strong foundations (and to prepare them for the mathematics they will encounter in the upper grades), it is essential that we do not encourage them to apply simplified rules in a superficial manner. By engaging in the types of activities described in this chapter, you can further support the development of your students' fraction sense and help them to understand fractions as numbers in which both the numerator and denominator need to be considered in order to understand their true value.

For More Information

For more information about the Math Pathways and Pitfalls program, see www.wested.org/cs/we/view/pj/81.

Study Questions

After reading Chapter 2:

1. What information presented in the "Classroom Scenario," "What's the Math?" and "What's the Research?" sections was familiar to you or similar to your experience with students?
2. What information presented in the "Classroom Scenario," "What's the Math?" and "What's the Research?" sections was new or surprising to you?
3. Which of the "Classroom Activities" ("Activity 2.1: Number Line Activities with Cuisenaire Rods"; "Activity 2.2: Is It Always True?"; and "Questions to Help Students Reason About Fractions as Numbers") do you plan to implement with your students?

After trying one or more of the activities:

1. Describe the activity and any modifications you made to meet your students' needs and/or to align with your curriculum.
2. How did this activity add to your knowledge of what your students do and do not understand about fractions as numbers?
3. What are your next steps for supporting your students' learning about fractions as numbers?