MESSAGE

31 Developing Mathematical Habits of Mind

LOOKING AT THE BACKGROUND, CONTEXT, AND CONTENT OF THE COMMON CORE STANDARDS FOR MATHEMATICAL PRACTICE

The only way to know mathematics is to do mathematics.

—Paul Halmos (*I Want to Be a Mathematician*, 1985)

In a fascinating glimpse into his mind, Paul Halmos (1985) describes his journey as a learner, teacher, and practitioner of mathematics. His description reflects a career of sometimes elegant and often messy exploration with mathematical ideas, properties, problems, and theorems, including wrong turns, dead ends, and unproductive thinking. His depiction of his life as a mathematician reflects the insights, *Aha!* moments, epiphanies, and serendipities that resulted from analyzing what worked and what didn't. He shares the value of interacting with colleagues, persevering through difficult paths that often required considerable amounts of time, and sometimes stepping away from a problem for a while. In looking at this image of what a "real" mathematician does, we can see the power of learning from mistakes and of backing up to reflect, consider, analyze, regroup, redirect, and move forward, building on curiosity and a desire to find solutions. What he describes in a very personal way are the *mathematical habits of mind* we want every student to develop.

What Are Mathematical Habits of Mind?

For years, mathematicians, educators, and other experts have tried to describe the heart of what it means to do mathematics and think mathematically, often using terms like *mathematical habits of mind*, *mathematical processes,* or *mathematical practices*. Students who learn only mathematical facts, definitions, rules, and procedures may do fine on large-scale tests that address these relatively easy-to-score elements of mathematics. But many of these same students later find that they cannot use what they know when they encounter any problem or situation they haven't specifically learned how to solve. On the one hand, we lament the poor preparation of students who can't apply what they've learned, but on the other hand, too often we continue to cling to the old notion that mathematics consists primarily of a checklist of knowledge and skills.

There is no one correct or complete list of mathematical habits of mind. Many descriptions overlap or address similar aspects of the nature of mathematics. (See the "More to Consider" section of this message for several descriptions and lists of mathematical habits of mind as conceptualized by various experts.) Almost all descriptions of mathematical habits of mind, mathematical thinking, practices, or processes center on a person's ability to solve mathematical problems, especially those that go beyond simple word problems related to a recently learned procedure.

Closely connected to solving problems is the ability to explain one's thinking and engage in productive discourse with others about the problem or observations about the mathematics in the problem. Thus, almost all discussions of mathematical habits of mind involve dimensions of thinking and reasoning. Some descriptions of mathematical habits of mind build from general intellectual habits of mind, such as perseverance, persistence, listening and communication skills, or metacognitive skills like reflection and analysis. Others may be uniquely associated with mathematics, such as considering multiple ways of representing mathematical ideas, zooming in and zooming out on particular aspects of a problem and on the problem as a whole, the ability to connect ideas within and outside of mathematics, making conjectures and generalizations, understanding the structure of mathematics, considering mathematical relationships, justifying and explaining mathematical solutions, and so on. These habits of mind span grade levels and ages; students can develop and demonstrate them in appropriate ways from their earliest experiences with mathematics. Given the right kinds of opportunities, a student's level of expertise in using mathematical habits of mind will increase year after year, ideally with students graduating from high school having developed a powerful set of mental abilities.

Connecting Mathematical Habits of Mind and the Common Core State Standards

The design of the Common Core State Standards includes both Standards for Mathematics Content and Standards for Mathematical Practice in acknowledgment of what mathematicians and mathematics educators have recognized for years—that it is not possible to be knowledgeable about mathematics if all a person knows is mathematical content. The essential partner to mathematical content is a set of mathematical ways of thinking and reasoning that can equip a person to navigate through hard or unknown mathematical territory.

The Common Core's descriptions of the Standards for Mathematical Practice address many, if not most, of the dimensions of mathematical thinking and habits of mind articulated in previously published discussions. Thus, considering these practices can give us a good, broad overview of the nature of mathematical habits of mind essential for today's students. In considering these practices, we should also keep in mind excellent recommendations from other sources in recent years, most notably the process standards from the National Council of Teachers and the mathematical proficiencies described in *Adding It Up* (National Research Council 2001). Both of these sources are acknowledged in the Common Core State Standards documents, and the writers have also considered other important discussions of mathematical habits of mind listed in the "More to Consider" section of this message, especially the work done by Al Cuoco, E. Paul Goldenberg, and June Mark (2010).

PRACTICES

The Common Core State Standards' explicit attention to mathematical habits of mind is represented by the Standards for Mathematical Practice (NGA Center and CCSSO 2010, 6). Increasingly these critical practices are being recognized as central to the goals of mathematics teaching and learning and many consider the practices the most central component of the standards. One vision of the Standards for Mathematical Practice groups the eight standards into four related pairs (see Figure 31.1).

In this organization of the eight Standards for Mathematical Practice, the four major types of mathematical practices are:

- reasoning and explaining,
- modeling and using tools,
- seeing structure and generalizing, and
- overarching habits of mind of a productive mathematical thinker, including problem solving and communication.

One of the important messages in this graphic is a reminder that the eight practices may not be as discrete as they initially appear. Rather, they function together, not only as pairs of standards, but as a cohesive set of descriptors contributing to our notions of what mathematical habits of mind we hope to help every student develop.

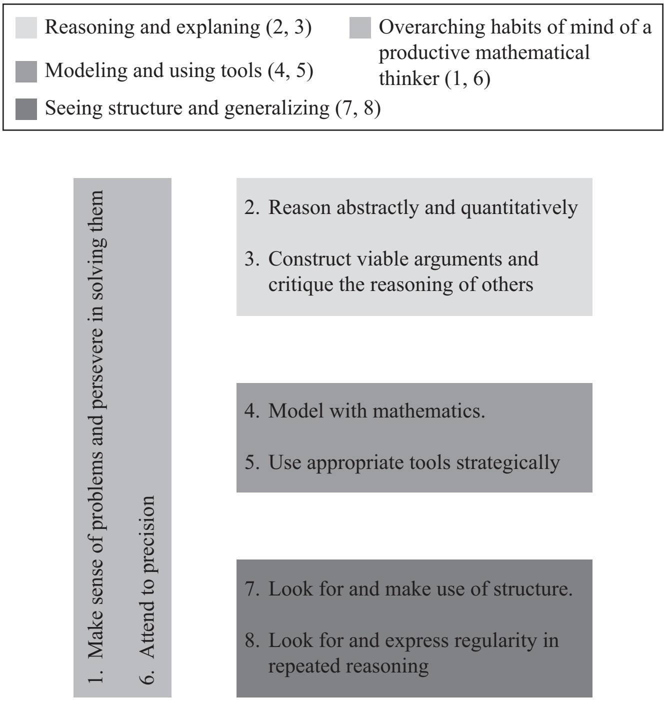

Figure 31.1

Common Core Eight Standards of Mathematical Practice

Source: http://commoncoretools.me/2011/03/10/structuring-the-mathematical-practices

PROCESSES

As we explore these practices, consider NCTM's standards for mathematical processes:

- Problem Solving
- Reasoning and Proof
- Communication
- Connections
- Representations

In thinking about how the eight CCSSM Practices relate to NCTM's five process standards, groups of experts and practitioners are likely to arrive at different ways of cross-matching the two sets of standards, demonstrating how overlapping and nondiscrete any list of mathematical habits of mind is likely to be. One model from *Connecting the*

NCTM Process Standards and the CCSSM Practices (Koestler, Felton, Bieda, and Otten 2013) shows the following matches:

NCTM Process Standards	CCSSM Practices
Problem Solving	1, 2, 4, 5
Reasoning and Proof	1, 3, 8
Communication	1, 2, 4, 6
Connections	1, 2, 4, 7, 8
Representations	1, 2, 4, 5, 6, 7

It's far less important to identify which particular standard(s) a given problem or practice addresses than it is to look for opportunities to focus on and help students develop one or more of the practices within the context of the problem. In fact, a valuable professional learning experience, especially among colleagues or within a professional learning community, can be to do a matching among these two sets of standards for mathematical habits of mind, considering the intent of each standard and what each standard seems to address most directly. My own matching differs somewhat from the one above, but agreeing on a list is much less important than the discussions that can arise as individuals and colleagues consider the standards in depth as related to their own work.

PROFICIENCY

The National Research Council's *Adding It Up* (2001) offers a vision of *mathematical proficiency* that echoes many of the same notions as the practices and processes described earlier. The NRC identifies five components describing what is necessary for a person to learn mathematics successfully:

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations;
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- *strategic competence*—ability to formulate, represent, and solve mathematical problems;
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification; and
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Looking at these strands of proficiency, the first two—conceptual understanding and procedural fluency—seem to address the kind of mathematics knowledge and skills most often represented in state mathematics standards. The last three strands—strategic competence, adaptive reasoning, and productive disposition—seem to reflect mathematical habits of mind related to solving problems, reasoning, justifying, and persistence and willingness to tackle mathematical problems, as well as confidence. As in other discussions of mathematical habits of mind, the NRC notes that these five strands are interwoven and interdependent, a notion they try to represent graphically using a rope metaphor (see www.nap.edu/openbook.php?record_id=9822&page=117).

Incorporating Mathematical Habits of Mind

Through in-depth consideration of the Common Core Standards for Mathematical Practice and these processes and strands of proficiency, educators can begin to determine how mathematical habits of mind might be developed in their particular schools and classrooms. Ideally, we can learn how to seamlessly incorporate these habits into our mathematics programs so that both teachers and their students come to routinely view mathematics as a rich, powerful, and useful set of thinking and analytical tools they can use to make sense of and tackle a wide variety of problems both in and outside of mathematics.

Many good mathematics tasks offer opportunities for students to use or develop mathematical habits of mind. Even some routine or procedural problems can offer opportunities for students to call on mathematical habits of mind if we ask appropriate questions to push students' thinking beyond an obvious or superficial response. Consider the following problem:

> *A store is advertising a sale with 10% off all items in the store. Sales tax is 5%. A 32-inch television is regularly priced at $295.00. What is the total price of the television, including sales tax, if it was purchased on sale?* (Charles A. Dana Center at The University of Texas at Austin 2012b)

This is a good applied problem—a fairly traditional word problem. To solve it requires a couple of steps, but the solution path is clear if a student understands percent, and the answer will either be right or wrong. Now consider the following extension of the same problem:

> *Adam and Brandi are customers discussing how the discount and tax will be calculated. Adam says that to find the total cost for any item in the store, you can take 10% off the original price, then add the sales tax to the discounted price. Brandi says that to find the total cost for any item in the store, you can determine the original price of the item, including sales tax, and*

then take 10% off. Are both Adam and Brandi correct? Justify your answer. (Charles A. Dana Center at The University of Texas at Austin 2012b)

The extended problem requires students to apply further mathematical reasoning, not just aiming at a numerical answer, but comparing two different procedures with subtle but important differences. Further, simply by asking students to justify their answer, we ramp up the thinking and reasoning involved.

Messages 32–39 in this section look more closely at each of the eight mathematical practices and urge us to consider how each might be addressed. Appendix A includes several sources for problems and tasks that allow for deep thinking, reflection, analysis, explanation, and reasoning, among other mathematical habits of mind.

Assessing Mathematical Habits of Mind

The fundamental idea of building mathematics programs grounded in mathematical habits of mind has been advocated for decades, but its actualization has sometimes eluded teachers, textbook authors, curriculum developers, standards writers, and test developers. Even though we can see the value in helping students develop ways of using mathematical thinking to make sense of their world and solve the many types of problems they will encounter, sometimes it simply seems too time consuming, expensive, or difficult to make real. In particular, we have not seen widespread use of appropriate assessments to support the teaching of mathematical thinking and habits of mind.

While the NCTM standards from both 1989 and 2000 included specific standards on mathematical processes, most state standards in the late 1990s and well into the twenty-first century consisted primarily of lists of mathematical content. If mathematical processes were addressed at all, they may have appeared in relatively invisible introductory paragraphs or accompanying narratives describing how important it was to incorporate problem solving, reasoning, and so on. Since state tests tended to focus on the standards themselves, rather than on the invisible paragraphs and narratives, rarely, if ever, did these dimensions of mathematical thinking appear on such tests.

Now, however, we see indications that the Common Core standards may bring with them the promise of a new era in assessment to support the mathematical thinking and habits of mind we value. In contrast to the widespread lack of attention on assessments in the past to mathematical habits of mind and mathematical processes, the two primary large-scale tests designed to accompany the standards have indicated a commitment to focus primarily on these practices, at least in rhetoric and intention (PARCC/SBAC). It may take several years for these tests and other state and local assessments to realize the full power of inclusion of these elements of mathematical thinking. While there are noteworthy examples of assessing mathematical thinking in the classroom and in some curriculum programs, this is new ground for large-scale testing. If high-stakes tests can measure and reward deep aspects of mathematical thinking, reasoning, and problem solving, perhaps teachers will feel that they're allowed to teach toward these habits of mind.

What Can We Do?

> Being a mathematician is no more definable as "knowing" a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts. . . . Being a mathematician, again like being a poet, or a composer or an engineer, means doing, rather than knowing or understanding.
>
> —Seymour Papert ("Teaching Children to Be Mathematicians Versus Teaching About Mathematics," 1972)

The work of mathematicians who *do* mathematics as Seymour Papert describes involves thinking, reasoning, looking for patterns, noticing and connecting elements of structure, and solving complex problems using mathematical tools, among other things. It's only when our students engage in actually doing mathematics—working on hard problems, engaging in discussion, arguing, explaining, interacting with mathematical ideas and paying attention to their thinking as they do so—that they come to know mathematics well and develop a positive disposition toward the subject. Becoming proficient at particular habits is not so much an end goal as a lifelong journey. Even professional mathematicians continue to hone and refine these habits throughout their career. And there is no profession where this kind of lifelong learning and growth is a higher priority than for teachers of mathematics, regardless of the level or age of students they teach. Teachers not only become better at helping students learn mathematics; they serve as powerful role models. If students are to develop mathematical habits of mind in ways that will serve them in the future, then we need to examine our curricula, assessments, and instruction in light of such habits of mind. We can also help students themselves become aware that the purpose of their mathematics learning is much more than the skills, facts, procedures, and even concepts they learn. When we do so, we not only improve students' understanding and proficiency, we also improve their attitudes toward mathematics and their interest in doing more of it.

The Common Core State Standards provide us with a unifying vehicle to help students develop the crucial habits of mind they need in order to learn mathematics well and, especially, in order to use what they learn after they leave school. The Standards for Mathematical Practice offer a new structure for understanding mathematical habits of mind. As we work to implement these eight practices, we need to be patient with ourselves and remember that the practices involve mathematically sophisticated ideas that are not always easy to understand, even for those with a mathematics background. It will take learning and effort to implement the practices well. Mathematics is a science of patterns, and we can look for patterns ourselves as we make sense of the Standards for Mathematical Practice. The patterns we find might take the form of common themes such as:

- looking for, articulating, and using patterns in mathematics (to make generalizations, to recognize mathematical structure, to solve problems, and so on);
- learning to reason and make sense of mathematics (reasoning takes many forms and crosses many practices);
- zooming out and zooming in (backing up to look at the big picture of a concept, problem, or connecting topic, and focusing back in on the specifics); and
- representing mathematical situations and ideas in many ways and moving back and forth between representations.

Mathematics is also held together by a web of connections. As we continue to learn about the practices and collaborate on how best to help students internalize them, we will discover that the practices are neither separate nor sequential. The practices blend together and overlap in beautiful and messy ways, sometimes confusing us about which practice we're seeing or using. We need to remember that such distinctions are contrary to the vision of the Standards for Mathematical Practice—the vision of every student possessing a unified and useful set of mathematical habits of mind. It's much better to keep our eye on the overall picture painted by the set of eight practices together, rather than keeping our eye on a checklist of which practices a student may or may not have mastered.

In our fast-paced, technology-driven world and competitive global workplace environment, today's students—tomorrow's workers—must be able to reason, think, and figure out how to approach and solve problems they've never specifically learned how to solve. If students leave school having learned mathematical content alone, without having learned these twenty-first-century survival skills, we will have woefully underprepared them for their future. Perhaps the time has come when enough people realize the importance of these powerful habits and when we have learned enough about how to help students develop them. Perhaps the time has come when we can finally garner the national will to actualize the goal of helping every student develop mathematical habits of mind that can serve them throughout their lives.

Reflections and Discussion

FOR TEACHERS

- What issues or challenges does this message raise for you? In what ways do you agree with or disagree with the main points of the message?
- When do you use mathematical habits of mind in your everyday life?
- How do you demonstrate mathematical habits of mind in your work with students?
- How familiar are you with the Common Core Standards for Mathematical Practice? Which practices do you find most challenging to understand or implement?

FOR FAMILIES

- What questions or issues does this message raise for you to discuss with your son or daughter, the teacher, or school leaders?
- How can you help your daughter or son understand that succeeding in mathematics involves more than learning facts and procedures—that it involves learning how to think and reason?
- How familiar are you with the Common Core State Standards for Mathematics (available online at corestandards.org), especially the set of eight Standards for Mathematical Practice? Which practices do you find most challenging to understand? Where can you go for help to make sense of any standards you may not fully understand?

FOR LEADERS AND POLICY MAKERS

- How does this message reinforce or challenge policies and decisions you have made or are considering?
- How familiar are you with the Common Core Standards for Mathematical Practice? Which practices do you find most challenging to understand or implement?
- How well does your curriculum address mathematical habits of mind?
- How can you support your teachers in balancing the teaching of mathematical content and the development of mathematical practices?

RELATED MESSAGES

Smarter Than We Think

- Messages 32 through 39 explore each of the Standards for Mathematical Practice from the Common Core State Standards, incorporating important ideas from NCTM's process standards and other sources.
- Message 40, "Mathematical Habits of Instruction," pulls together ideas from this message and the other messages in Part IV, as well as drawing from messages throughout the book, to suggest how we can implement what we know to help students develop their abilities to think mathematically.
- Message 1, "Smarter Than We Think," reminds us of the importance of mathematical thinking for all students and, based on a growth mindset, emphasizes the role that challenges can play in helping students improve their intelligence, develop mathematically, and learn to think, reason, and make sense of mathematics.

Faster Isn't Smarter

- Message 14, "Balance Is Basic," makes a case for teaching a balanced program of knowledge, skills, understanding, and, most of all, mathematical thinking.
- Message 1, "Math for a Flattening World," considers the rapidly changing workplace and world around us in terms of the need to help individuals learn how to reason, think creatively, and solve problems we don't know the answers to.
- Message 3, "Making the Case for Creativity," emphasizes the importance of teaching creativity as part of a broader vision of mathematical thinking and reasoning.

MORE TO CONSIDER

- *I Want to Be a Mathematician* (Halmos 1985), a description of Halmos's life's work in mathematics, includes his wrong turns and approaches that didn't work out and offers an intimate view of what it means to be a mathematician, to do mathematics, and to think mathematically.
- "Implementing the Mathematical Practice Standards" (Education Development Center) gives background and overview of the CCSS practices and resources for classroom lessons, including both online resources and professional development. http://mathpractices.edc.org.
- "Mathematical Practice Institute" (Education Development Center) is a professional development institute for high school teachers focused on the mathematical practices. https://mpi.edc.org.

- Common Core State Standards: A New Foundation for Student Success: *The Importance of Mathematical Practices* (McCallum and Zimba 2011) is a four-minute video by Bill McCallum and Jason Zimba on the Standards for Mathematical Practice of the Common Core State Standards.
- "From the Inside Out" (Fillingim and Barlow 2010) describes the kind of mathematical thinking involved in helping children become "doers of mathematics" in and outside of school.
- "Mathematics, Mathematicians, and Mathematics Education" (Bass 2005) shares insights from a mathematician about priorities in school mathematics teaching, including the importance of mathematical thinking, and describes the role of mathematicians in working collaboratively to support those priorities.
- *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School* (Carpenter, Franke, and Levi 2003) offers background and strategies on how to focus elementary mathematics instruction on mathematical habits of mind that support the transition from numbers to symbols.
- *Learning and Leading with Habits of Mind: 16 Essential Characteristics for Success* (Costa and Kallick 2008) looks at productive habits of mind in general (not specifically related to mathematics), including personal behaviors and intellectual habits, and offers steps for educators on helping students develop habits of mind.
- "Contemporary Curriculum Issues: Organizing a Curriculum Around Mathematical Habits of Mind" (Cuoco, Goldenberg, and Mark 2010) suggests using mathematical habits of mind, rather than content topics, as a way to organize a mathematics program.
- "A Collection of Lists of Mathematical Habits of Mind" (Lim 2013) is a list of bullet points from several sources addressing aspects of mathematical habits of mind or general habits of mind.
- *Adding It Up: Helping Children Learn Mathematics* (National Research Council 2001) reports research around a conceptual definition of mathematical proficiency (and accompanying "rope" model), incorporating mathematical habits of mind, as cited in this message.
- *Connecting the NCTM Process Standards and the CCSSM Practices* (Koestler, Felton, Bieda, and Otten 2013) unpacks each practice and relates it to NCTM's five process standards, including sample classroom vignettes for elementary, middle, and high school.
- "Growth Mindset and the Common Core Math Standards" (Bryant 2013) looks at a growth mindset as it relates to students developing mathematical habits of mind described in the Common

Core Standards for Mathematical Practice. www.edutopia.org/blog/growth-mindset-common-core-math-cindy-bryant.

- "What Is Mathematics? A Pedagogical Answer to a Philosophical Question" (Harel in Gold and Simons 2008) discusses mathematical habits of mind as a central part of the discipline of mathematics and is influential as background for the Common Core Standards for Mathematical Practice (NGA Center and CCSSO 2010).
- *Mathematics and Plausible Reasoning, Volume II: Patterns of Plausible Inference* (Polya 2009) discusses multiple dimensions of mathematical thinking and reasoning, generally at the high school level, from the widely respected expert on problem solving.
- *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) offered a description of mathematical process standards as part of the first set of mathematics standards offered from the profession.
- *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000) describes mathematical content and process standards.
- Common Core State Standards for Mathematics (NGA Center and CCSSO 2010) includes descriptions of the eight Standards for Mathematical Practice.

WWW This message is also available in printable format at mathsolutions.com/smarterthanwethink.